

A polynomial action for gravity with matter, gauge-fixing and ghosts

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Abstract

We give a variational formulation of General Relativity, with coupling to a cosmological constant, to scalar fields, to vector fields and to spinor fields (all with possible mass and interaction terms). Among the matter fields, we include ghosts corresponding to diffeomorphism and Yang-Mills gauge symmetries, with kinetic terms given by gauge fixing conditions leading to hyperbolic equations of motion for all fields. The distinguishing characteristic of our Lagrangian density is its polynomiality, in all dynamical fields and Lagrange multipliers, and its validity for any number of spacetime dimensions.

Keywords: general relativity, variational principle, polynomial, Einstein-Palatini, scalar field, vector field, spinor field.

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Since the introduction of the Einstein-Hilbert variational principle for the equations of General Relativity, many alternative Lagrangians have been proposed, based on different choices of variables and with different algebraic or geometric structure. As long as an alternative Lagrangian can be reduced to the Einstein-Hilbert one by an algebraic redefinition or elimination of variables (that is, not involving the resolution of any differential equations), the alternative choice is largely equivalent to the original one. However, these alternative formulations may be useful for particular purposes, like development of geometric intuition, simplification of some calculations, non-linear deformation or coupling to other fields.

The Einstein-Hilbert Lagrangian (an n -form on an n -dimensional manifold) [6],

$$L_{\underline{a}}[g] = v_{\underline{a}}[g] g^{bc} R_{bc}[g], \quad (1)$$

is a function of the metric tensor g_{bc} , with g^{bc} the inverse metric, $v_{\underline{a}}[g]$ the metric volume form and $R_{bc}[g]$ the Ricci tensor. We are using a compressed index notation for n -forms, where $v_{\underline{a}} = v_{a_1 \dots a_n}$. When compared to other Lagrangians used in fundamental physics, like the scalar (Klein-Gordon), vector (Maxwell or Yang-Mills), spinor (Dirac), etc. Lagrangians, a striking distinction of the Einstein-Hilbert Lagrangian is that it is non-polynomial in the metric g_{bc} , its dynamical field.

While not essential from the physical point of view, polynomiality is a convenient technical property. The main simplification it brings is at the level of perturbation theory, either classical or quantum. When writing the metric $g = \bar{g} + \varepsilon h_1 + \varepsilon^2 h_2 + \dots$ as an additive perturbation of a background metric \bar{g} , the equations for g give rise to a hierarchical family of linear equations for each of the h_i , with inhomogeneous sources depending polynomially on the $h_{j < i}$. For equations obtained from a non-polynomial Lagrangian like (1), at each order of perturbation theory, the order of the polynomial dependence of the inhomogeneous sources on the $h_{j < i}$ terms will increase. This polynomial order stays bounded only if the original Lagrangian was itself polynomial. In perturbative quantum field theory, we can analogously say that a non-polynomial Lagrangian gives rise to new and higher order *Feynman diagram vertices* [10] at each perturbative order. Thus, the polynomiality of the Lagrangian may significantly simplify the algebraic structure of perturbation theory, at least at higher orders.

A polynomial Lagrangian obviously gives rise to polynomial Euler-Lagrange equations. Another advantage of polynomiality shows itself at the level of the formal analysis of Euler-Lagrange equations as partial differential equations. Any partial differential equation can be geometrically and invariantly represented as an equation on the space of jets of the dynamical fields (the dependent variables) or,

equivalently, the subspace of the jet space given by the vanishing locus of these equations [8]. When the equations are polynomial, this locus is an algebraic subvariety of the jet space, the *PDE subvariety*. In the general non-polynomial case, it is at best a smooth manifold or only a stratified manifold, when singularities are present. Understanding the geometry of the PDE subvariety is crucial for identifying symmetries and conservation laws, identifying integrability conditions, prolonging the equations to involution, analyzing the singularities of solutions, etc. Thus, the powerful machinery of algebraic geometry, dedicated to the analysis of the geometric properties of algebraic varieties, may be brought to bear on these questions.

So, it is a natural question to ask whether General Relativity can be formulated with a polynomial Lagrangian. For pure vacuum General Relativity, the answer is Yes and one such formulation is provided by the cubic Einstein-Palatini formulation [2, 7, 1]

$$L_{\underline{a}}[\mathbf{g}, C] = \mathbf{g}_{\underline{a}}^{bc}(\bar{R}_{bc} + R_{bc}[C]), \quad \text{with} \quad R_{bc}[C] = -\bar{\nabla}_b C_{dc}^d + \bar{\nabla}_d C_{bc}^d + C_{bc}^d C_d - C_{bd}^e C_{ce}^d, \quad (2)$$

where we should interpret the inverse densitized metric $\mathbf{g}_{\underline{a}}^{ab} = v_{\underline{a}}[g]g^{bc}$ as a fundamental variable, along with the Christoffel tensor C_{bc}^d , which parametrizes the difference between a connection ∇ and the Levi-Civita connection $\bar{\nabla}$ for a given background metric \bar{g} . Thus, \bar{R}_{bc} is the Ricci tensor of \bar{g} and $\bar{R}_{bc} + R_{bc}[C]$ becomes equal to the Ricci tensor for g , once C_{bc}^d assumes the standard Levi-Civita form for g . Since the variation with respect to C yields an equation for C equivalent to $\nabla_d g_{bc} = 0$, eliminating the auxiliary Christoffel tensor from the Lagrangian gives back the standard Einstein-Hilbert form, though written in terms of \mathbf{g} and thus commonly known as the Goldberg Lagrangian [3].

Unfortunately, it is not possible to maintain polynomial form with the same variables as above when expanding the Lagrangian by including a cosmological constant term or couplings to matter fields, both of which are of interest in physical applications. The challenge then becomes to introduce new auxiliary fields, like Lagrange multipliers, in terms of which a sufficiently rich Lagrangian may be written in polynomial form. We will content ourselves with adding (a) a cosmological constant, (b) a massive scalar field, (c) a (Maxwell or Yang-Mills) gauge field, (d) a Dirac spinor field, and (e) gauge-fixing and corresponding ghost terms for those fields that have gauge symmetries. Point (e) is a necessary ingredient for a consistent covariant quantization of this field theory [4]. Below, we give the final result, with some explanations:

The field content is $\Phi = (\mathbf{g}, C, B, u, \bar{u}, v, w, \phi, \hat{A}, \hat{F}, \hat{b}, \hat{z}, \hat{\bar{z}}, e, f, S, T, \psi)$, with its interpretation below. The hatted fields are Lie algebra valued. The Lie algebra is semi-simple, with commutator $[-, -]$ and invariant positive definite inner product $\langle -, - \rangle$. The real scalar and complex spinor multiplets carry representations of the Lie algebra, denoted $(-) \cdot \phi$ or $(-) \cdot \psi$, and invariant inner products $\langle -, - \rangle$, with $\langle \phi, A \cdot \phi \rangle = -\langle A \cdot \phi, \phi \rangle$ and $\langle \psi, A \cdot \psi \rangle = \langle A \cdot \psi, \psi \rangle$. We denote by $\bar{\gamma}^a$ a given choice of hermitian γ -matrices for the given background metric \bar{g}_{bc} , with $\bar{\nabla}$ its Levi-Civita connection, extended to spinors such that $\bar{\nabla}_b \bar{\gamma}^c = 0$.

$\mathbf{g}_{\underline{a}}^{bc}$: inverse densitized metric,	\hat{A}_b : gauge vector field,
$C_{(bc)}^a$: Christoffel tensor,	\hat{F}_b^c : gauge curvature tensor,
B_c : de Donder Nakanishi-Lautrup field,	\hat{b} : Lorenz Nakanishi-Lautrup field,
u^b, \bar{u}_c : diffeomorphism ghost and anti-ghost,	$\hat{z}, \hat{\bar{z}}$: gauge ghost and anti-ghost,
ϕ : scalar field (multiplet),	ψ : spinor (multiplet),
$v_{\underline{a}}$: metric volume form,	$w^{\underline{k}^4 \dots \underline{k}^n}$: its Lagrange multiplier,
$e_{(bc)}$: Lorentz frame field,	$f^{(bc)}$: its Lagrange multiplier,
T_{bcd} : spin torsion,	S^{bcd} : its Lagrange multiplier.

$$\begin{aligned}
L_{\underline{a}}[\Phi] = & \underbrace{\mathfrak{g}_{\underline{a}}^{bc}(\bar{R}_{bc} + R_{bc}[C])}_{(a1):3} - \underbrace{B_b(\mathfrak{g}_{\underline{a}}^{bc}B_c - 2\bar{\nabla}_c\mathfrak{g}_{\underline{a}}^{bc})}_{(a2):3} - \underbrace{2\bar{\nabla}_{(b}\bar{u}_{c)}(\mathcal{L}_u\mathfrak{g}_{\underline{a}})^{bc}}_{(a3):3} \\
& + \underbrace{w^{\underline{k}^4}\dots\mathfrak{g}_{\underline{a}}^{b_1\dots b_n}(G_{\underline{a}k^4\dots k^n}[\mathfrak{g}] + n!v_{\underline{a}}v_{k^4}\dots v_{k^n})}_{(b0):n+1} - \underbrace{\frac{1}{2}(\mathfrak{g}_{\underline{a}}^{bc}\langle(d\phi)_b, (d\phi)_c\rangle + m_\phi^2\langle\phi, \phi\rangle v_{\underline{a}})}_{(b1):3} \\
& + \underbrace{\frac{1}{4}\langle\hat{F}_b^c, v_{\underline{a}}\hat{F}_c^b - 2\mathfrak{g}_{\underline{a}}^{bd}(D\hat{A})_{cd}\rangle}_{(c1):3+1} + \underbrace{\frac{1}{2}\langle\hat{b}, v_{\underline{a}}\hat{b} - 2\mathfrak{g}_{\underline{a}}^{bc}\bar{\nabla}_b\hat{A}_c\rangle}_{(c2):3} + \underbrace{\frac{1}{2}\langle\bar{\nabla}_b(\mathfrak{g}_{\underline{a}}^{bc}\hat{z}), (D\hat{z})_c\rangle}_{(c3):3+1} \\
& + \underbrace{f^{de}(\mathfrak{g}_{\underline{a}}^{bc}e_{bd}e_{ce} - v_{\underline{a}}\bar{g}_{de}) + S^{bcd}[v_{\underline{a}}e_{cc'}\bar{g}^{c'c''}T_{bc''d} - (e_{fb}\bar{\nabla}_ce_{de})\mathfrak{g}_{\underline{a}}^{ef}]}_{(d0):4} \\
& - \underbrace{\frac{1}{2}\mathfrak{g}_{\underline{a}}^{bd}e_{bc}(\langle\psi, i\bar{\gamma}^c\bar{\nabla}_d\psi\rangle + \langle i\bar{\gamma}^c\bar{\nabla}_d\psi, \psi\rangle) - \frac{1}{4}v_{\underline{a}}\langle\psi, i\bar{\gamma}^{[b}\bar{\gamma}^c\bar{\gamma}^{d]}T_{bcd}\psi\rangle - m_\psi\langle\psi, \psi\rangle v_{\underline{a}}}_{(d1):4} \\
& - \underbrace{2\Lambda v_{\underline{a}}}_{(aa):1} - \underbrace{\lambda\phi^4 v_{\underline{a}}}_{(bb):5} - \underbrace{\frac{m_A^2}{2}\mathfrak{g}_{\underline{a}}^{bc}\langle\hat{A}_b, \hat{A}_c\rangle}_{(cc):3} + \underbrace{\frac{q_\phi}{2}\mathfrak{g}_{\underline{a}}^{bc}(\langle\bar{\nabla}_c\phi, \hat{A}_b\cdot\phi\rangle + \langle\hat{A}_c\cdot\phi, \bar{\nabla}_b\phi\rangle) + \frac{q_\phi^2}{2}\mathfrak{g}_{\underline{a}}^{bc}\langle\hat{A}_b\cdot\phi, \hat{A}_c\cdot\phi\rangle}_{(bc):5} \\
& - \underbrace{\mu\phi\langle\psi, \psi\rangle v_{\underline{a}}}_{(bd):4} - \underbrace{\frac{q_\psi}{2}\mathfrak{g}_{\underline{a}}^{bd}e_{bc}(\langle\psi, \bar{\gamma}^c\hat{A}_d\cdot\psi\rangle + \langle\bar{\gamma}^c\hat{A}_d\cdot\psi, \psi\rangle)}_{(cd):5}, \quad (3)
\end{aligned}$$

where the terms are labeled by their role and polynomial degree, and we have also used

$$R_{bc}[C] = -\bar{\nabla}_b C_{dc}^d + \bar{\nabla}_d C_{bc}^d + C_{bc}^d C_d - C_{bd}^e C_{ce}^d, \quad (4)$$

$$(\mathcal{L}_u\mathfrak{g}_{\underline{a}})^{bc} = u^d\bar{\nabla}_d\mathfrak{g}_{\underline{a}}^{bc} - 2\mathfrak{g}_{\underline{a}}^{d(b}\bar{\nabla}_{d}u^{c)} + \mathfrak{g}_{\underline{a}}^{bc}\bar{\nabla}_d u^d, \quad (5)$$

$$G_{\underline{a}k^4\dots k^n}[\mathfrak{g}] = \mathfrak{g}_{\underline{a}}^{b_1c_1}\mathfrak{g}_{\underline{b}}^{b_2c_2}\mathfrak{g}_{\underline{c}}^{b_3c_3}\mathfrak{g}_{\underline{k}^4}^{b_4c_4}\dots\mathfrak{g}_{\underline{k}^n}^{b_nc_n} \quad (\text{recalling } \underline{b} = b_1\dots b_n, \underline{c} = c_1\dots c_n), \quad (6)$$

$$(D\hat{A})_{dc} = (d\hat{A})_{dc} + [\hat{A}_d, \hat{A}_c], \quad (7)$$

$$(D\hat{z})_c = (d\hat{z})_c + [\hat{A}_c, \hat{z}]. \quad (8)$$

When \hat{A} is valued in an abelian Lie algebra, the polynomial degrees of (c1), (c3) drop by 1.

The roles of the various terms are as follows.

(a1): gravity kinetic term,	(c1): gauge kinetic term,
(a2): de Donder gauge-fixing term,	(c2): Lorenz gauge-fixing term,
(a3): diffeomorphism ghost kinetic term,	(c3): gauge ghost kinetic term,
(b0): auxiliary fields for scalars,	(d0): auxiliary fields for spinors,
(b1): scalar kinetic term with mass,	(d1): spinor kinetic term with mass,
(aa): cosmological constant term,	(bc): scalar-gauge coupling,
(bb): scalar potential,	(bd): (Yukawa) scalar-spinor coupling,
(cc): (Proca) vector mass term,	(cd): gauge-spinor coupling.

It should be noted that the (Proca) vector mass term with $m_A^2 \neq 0$ is incompatible with the vector gauge invariance. So, for consistency, either only (cc) or only the (c2) and (c3) terms should be included. The scalar and spinor masses, m_ϕ^2 and m_ψ , the scalar coupling, λ , the scalar and spinor charges, q_ϕ and q_ψ , and the Yukawa coupling, μ , should be thought of as tensors with respect to the appropriate multiplet structures.

Upon eliminating all the auxiliary fields (those that can be eliminated by solving their Euler-Lagrange equations algebraically), the resulting Lagrangian is the standard Lagrangian for General Relativity coupled to a Standard Model-like matter theory. The overall polynomial degree of the Lagrangian is $\max\{5, n+1\}$, with the dimension dependent degree coming from the definition of the auxiliary volume form field, $v_{\underline{a}}$. The above form is far from the only way of putting a similar Lagrangian in polynomial form (cf. [9, 5]). It is interesting to consider whether a polynomial form could be achieved with a smaller number of auxiliary fields or smaller polynomial degree in various terms.

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